Conserved quantities in the perturbed Friedmann world model

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The evolutions of linear structures in a spatially homogeneous and isotropic world model are characterized by some conserved quantities: the amplitude of gravitational wave is conserved in the super-horizon scale, the perturbed three-space curvature in the comoving gauge is conserved in the super-sound-horizon scale, and the angular momentum of rotational perturbation is generally conserved.

1. Summary: We consider a spatially homogeneous and isotropic world model with the most general, spacetime dependent, three (scalar, vector, and tensor) types of linear structures. As the gravity sector we consider both Einstein gravity with a hydrodynamic fluid or a scalar field, and a class of generalized versions of gravity which couples the scalar field with the scalar curvature. The three types of structures decouple from each other due to the symmetry in the background world model and the linearity of the structures we are assuming. We identify the generally conserved quantities existing in all three types of perturbations, ignoring the imperfect fluid source terms:

(A) In a near flat background the scalar-type (related to the density condensation) structure is characterized by a conserved quantity in the large-scale which is the perturbed three-space curvature in the comoving (or the uniform-field) gauge. This quantity is conserved in the super-sound-horizon scale, and thus is conserved effectively in all scales in the matter dominated era.

(B) The vector-type (rotation) structure is characterized by the angular momentum conservation.

(C) In a near flat (vanishing spatial curvature) background the amplitude of the tensor-type (gravitational wave) structure is conserved in the super-horizon scale.

In a similar sense, the sign of three-space curvature in the underlying world model, K, can be also regarded as a conserved quantity.

In the fluid and the scalar field dominated eras the conserved properties remain valid independently of changing (i.e., time-varying) equation of state $p(\mu)$ and changing field potential $V(\phi)$, respectively. These conservation properties also apply in a class of generalized versions of gravity theories including the Brans-Dicke theory, the non-minimally coupled scalar field, the scalar-curvature-square gravity, the low-energy effective action of the string theory, etc. As long as the conditions are met the conservation properties remain valid independently of changing gravity theories. The conservation properties in (B,C) are valid in the multi-component situations, whereas (A) applies to single-component situations.

In the following we will present the above mentioned results by concretely showing the equations describing the results. Since this is a summary of our previous works, the details about derivations and explanations are referred to the original works in the literature.

2. Gravity theories: We consider gravity theories belonging to the following action

$$S = \int d^4 x \sqrt{-g} \Big[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^{;a} \phi_{,a} - V(\phi) + L_m \Big], \tag{1}$$

where ϕ and R are the scalar field and the scalar curvature, respectively. L_m is the hydrodynamic part of the Lagrangian with the hydrodynamic energy-momentum tensor T_{ab} defined as $\delta(\sqrt{-g} L_m) \equiv \frac{1}{2}\sqrt{-g} T^{ab} \delta g_{ab}$.

3. Cosmological perturbations: We consider spacetime dependent perturbations in the background Friedmann world model

$$\begin{aligned} ds^2 = & - (1 + 2\alpha)\, dt^2 - a\,(\beta_{,\alpha} + B_\alpha)\, dt dx^\alpha \\ & + a^2 \Big[g^{(3)}_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{|\alpha\beta} \\ & + 2 C_{(\alpha|\beta)} + 2 C_{\alpha\beta} \Big] dx^\alpha dx^\beta. \end{aligned} \tag{2}$$

α, β, γ, and φ indicate the scalar-type structure with four degrees of freedom. The transverse B_α and C_α indicate the vector-type structure with four degrees of freedom. The transverse-tracefree $C_{\alpha\beta}$ indicates the tensor-type structure with two degrees of freedom. The three types of structures are related to the density condensation, the rotation, and the gravitational wave, respectively. Since these three types of structures evolve independently, we will handle them separately.

We also consider the general perturbations in the hydrodynamic energy-momentum tensor and the scalar field:

$$\begin{aligned} T_{ab}(\mathbf{x}, t) &= \bar{T}_{ab}(t) + \delta T_{ab}(\mathbf{x}, t), \\ \phi(\mathbf{x}, t) &= \bar{\phi}(t) + \delta\phi(\mathbf{x}, t). \end{aligned} \tag{3}$$

4. Gauge issue: We have ten degrees of freedom in the metric perturbation. Two (one temporal and one spatial) degrees of freedom in the scalar-type perturbation,

and two (both spatial) degrees of freedom in the vector-type perturbation are due to the spacetime coordinate (gauge) transformation. In order to handle these fictitious degrees of freedom we have the right to impose four conditions on the respective perturbed metric or energy-momentum content. Due to the spatial symmetry (homogeneity) the three spatial degrees of freedom can be uniquely fixed, thus making the remaining variables spatially gauge-invariant [1]; for example, $\chi \equiv a(\beta + a\dot{\gamma})$ is a spatially gauge-invariant combination. After removing the spatial gauge degrees of freedom, only the scalar-type perturbed variables are affected by the temporal gauge condition. When we fix the temporal gauge condition, we have *several* meaningful choices. Since usually we do not know which gauge condition will turn out to be most useful for the problem *a priori*, it is advantageous to start from equations without fixing the temporal gauge condition, thus in a gauge-ready form, [1,2].

Using the gauge-ready approach we have investigated the cosmological perturbation based on the hydrodynamic fluid, scalar field, and a class of generalized gravity, [2–6]. From these studies we find the best conserved quantity is the perturbed three-space curvature φ in the comoving gauge ($v \equiv 0$) or in the uniform-field gauge ($\delta\phi \equiv 0$); in the generalized gravity theory the uniform-field gauge is the better gauge condition, and the uniform-field gauge coincides with the comoving gauge in the minimally coupled scalar field. Both gauge conditions completely fix the gauge transformation and each variable evaluated in the gauge is the same as the corresponding unique gauge invariant combination of the variable and a variable used in the gauge condition. Since we have many different gauge conditions we proposed to write the gauge-invariant combinations in the following way, [2]:

$$\varphi_v \equiv \varphi - \frac{aH}{k} v, \quad \varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}} \delta\phi \equiv -\frac{H}{\dot{\phi}} \delta\phi_\varphi, \tag{4}$$

where k is a comoving wavenumber and $H \equiv \dot{a}/a$. φ_v is the same as φ in the comoving gauge condition which takes $v \equiv 0$, etc; v is a velocity related perturbed variable, [7].

5. Evolution equations: The equations for background are:

$$H^2 = \frac{1}{3F} \left[\mu + \frac{1}{2} \left(\omega\dot{\phi}^2 - f + RF + 2V \right) - 3H\dot{F} \right] - \frac{K}{a^2}, \tag{5}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{2\omega} \left(\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 2V_{,\phi} \right) = 0, \tag{6}$$

$$\dot{\mu} + 3H(\mu + p) = 0, \tag{7}$$

where $F \equiv \partial f / (\partial R)$. K is the sign of the background spatial curvature. It can be considered as an integration constant, and thus is a conserved quantity for the given world model.

In the following we summarize the evolution equations for three types of perturbations. We consider *near flat* background, thus neglect K term.

(A) In handling the scalar-type structure the proper choice of the gauge condition simplifies the analyses and the resulting equations. A gauge invariant combination of the perturbed curvature variable based on the comoving (or uniform-field) gauge is found to be the best conserved quantity under various changes. The equations describing the evolutions of the hydrodynamic [7], the minimally coupled scalar field [8], and the generalized gravity theories [9], respecively, are the following:

$$\frac{c_s^2 H^2}{a^3(\mu+p)} \left[\frac{a^3(\mu+p)}{c_s^2 H^2} \dot{\varphi}_v \right]^{\cdot} - c_s^2 \frac{\Delta}{a^2} \varphi_v = \text{stresses}, \tag{8}$$

$$\frac{H^2}{a^3\dot{\phi}^2} \left(\frac{a^3\dot{\phi}^2}{H^2} \dot{\varphi}_{\delta\phi} \right)^{\cdot} - \frac{\Delta}{a^2} \varphi_{\delta\phi} = 0, \tag{9}$$

$$\frac{\left(H + \frac{\dot{F}}{2F} \right)^2}{a^3 \left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F} \right)} \left[\frac{a^3 \left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F} \right)}{\left(H + \frac{\dot{F}}{2F} \right)^2} \dot{\varphi}_{\delta\phi} \right]^{\cdot} - \frac{\Delta}{a^2} \varphi_{\delta\phi} = 0, \tag{10}$$

where Δ is a Laplacian operator based on $g^{(3)}_{\alpha\beta}$ and $c_s^2 \equiv \dot{p}/\dot{\mu}$. For a pressureless ideal fluid, instead of Eq. (8) we have $\dot{\varphi}_v = 0$.

(B) We introduce $B_\alpha \equiv bY^{(v)}_\alpha$ and $C_\alpha \equiv cY^{(v)}_\alpha$ where $Y^{(v)}_\alpha$ is a (transverse) vector harmonic function, [10,11]. $v_\omega \equiv v^{(v)} - b$ is a gauge invariant combination related to the amplitude of hydrodynamic vorticity ω as $v_\omega \propto a\omega$, [11]. The rotational structure is described by

$$\left[a^4 (\mu + p) v_\omega \right]^{\cdot} = \text{stress}. \tag{11}$$

Thus, neither the generalized nature of the gravity theory nor the presence of scalar field affects the rotational perturbation in the hydrodynamic part. We note that Eq. (11) is valid even in a generalized gravity with the Ricci-curvature-square term in the action, [11].

(C) The gravitational wave is described by [2,12]

$$\frac{1}{a^3 F} \left(a^3 F \dot{C}^\alpha_\beta \right)^{\cdot} - \frac{\Delta}{a^2} C^\alpha_\beta = \text{stress}. \tag{12}$$

6. Solutions: Equations (8-12) immediately lead to general solutions in certain limiting situations.

(A) From Eqs. (8-10) we have the large-scale solution, for the hydrodynamic (ignoring the stresses), the scalar field, and the generalized gravity:

$$\varphi_v = C(\mathbf{x}) - \tilde{D}(\mathbf{x}) \int_0^t \frac{c_s^2 H^2}{a^3(\mu+p)} dt, \tag{13}$$

$$\varphi_{\delta\phi} = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t \frac{H^2}{a^3\dot{\phi}^2} dt, \tag{14}$$

$$\varphi_{\delta\phi} = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t \frac{\left(H + \frac{\dot{F}}{2F}\right)^2}{a^3 \left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}\right)} dt, \qquad (15)$$

where C and D (or $\tilde{D}$) are integration constants indicating coefficients of relatively growing and decaying solutions, respectively. Compared with the solutions in the other gauge conditions, the decaying modes in these solutions are *higher order* in the large-scale expansion, [6,7]. Thus, ignoring the transient (and also subdominating in the large-scale) modes we have the conserved quantity

$$\varphi_{\delta\phi} = C(\mathbf{x}) = \varphi_v. \qquad (16)$$

From Eqs. (8-10) we notice that Eq. (13) is valid in the super-sound horizon scale, whereas Eqs. (14,15) are valid in the super-horizon scale.

In the large-scale limit φ in many different gauge conditions shows the conserved behavior: in the case of an ideal fluid see Eqs. (41,73) in [3], and Eq. (34,35) in [4]; in the case of the scalar field see Eqs. (92) in [5]; and in the case of the generalized gravity see Sec. VI in [6]. The often discussed conserved variable ζ introduced in [13] is φ_δ which is φ in the uniform-density gauge. In [7] we made arguments why we regard φ_v as the best conserved quantity. Conservation properties are further discussed in [14].

(B) For vanishing anisotropic stress, Eq. (11) for the rotation mode has a solution [15,11]

$$a^3(\mu + p) \cdot a \cdot v_\omega \sim L(\mathbf{x}). \qquad (17)$$

Thus, the rotation mode of the hydrodynamic part is characterized by a conservation of the angular momentum L.

(C) In the large-scale limit (super-horizon scale), ignoring the anisotropic stress, from Eq. (12) we have a general solution for the gravitational wave [16]

$$C^\alpha_\beta(\mathbf{x}, t) = c^\alpha_\beta(\mathbf{x}) - d^\alpha_\beta(\mathbf{x}) \int_0^t \frac{1}{a^3 F} dt, \qquad (18)$$

where c^α_β and d^α_β are integration constants indicating coefficients of relatively growing and decaying solutions, respectively. Ignoring the transient mode, the amplitude of gravitational wave in the super-horizon scale is temporally conserved.

7. Unified forms: The equations and the large-scale solutions for the scalar- and tensor-type structures can be written in unified forms as:

$$\frac{1}{a^3 Q}(a^3 Q \dot{\Phi})^{\cdot} - c_A^2 \frac{\Delta}{a^2}\Phi = 0, \qquad (19)$$

$$\Phi = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t (a^3 Q)^{-1} dt, \qquad (20)$$

where, for the scalar-type fluid, the generalized gravity, and the tensor-type structures, respectively, we have:

$$\Phi = \varphi_v, \quad Q = \frac{\mu + p}{c_s^2 H^2}, \quad c_A^2 \to c_s^2, \qquad (21)$$

$$\Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + 3\dot{F}^2/2F}{\left(H + \dot{F}/2F\right)^2}, \quad c_A^2 \to 1, \qquad (22)$$

$$\Phi = C^\alpha_\beta, \quad Q = F, \quad c_A^2 \to 1. \qquad (23)$$

8. Applications: We have shown conserved quantities in all three types of structures in the Friedmann world model. Even in the super-horizon scale, not every quantity is conserved, and rather there exist some special (gauge-invariant) variables which are conserved independently of changing equation of state $p(\mu)$, changing scalar field potential $V(\phi)$, and changing gravity theories $f(\phi, R)$. Using the conserved quantities, as long as the linearity assumption is valid, we can easily trace the evolution of structures from the recent era to the early stage.

Let us consider a scenario where a generalized gravity (or Einstein gravity with a scalar field) dominates the early evolution stage of our observable patch of the universe, and at some point Einstein gravity takes over the dominance till the present era. If we further assume that the generalized gravity era provides an accelerated expansion (inflation) stage, the observationally relevant scales may exit the horizon to become super-horizon scales during the era. Under some conditions we can derive the generated quantum fluctuations based on the vacuum expectation value, and as the scale becomes the super-horizon it can be interpreted as the classical fluctuations based on the spatial average. As long as the relevant scales remain in the super-horizon stage during the transit epoch of the gravities (or changing potential, or changing equation of state) there exist *conserved quantities*: C^α_β and φ_v (or $\varphi_{\delta\phi}$) are the conserved quantities.

Now, let us explain more concretely how the conserved quantities provide connections between the hydrodynamic perturbations in Newtonian regime and the quantum fluctuations in the early universe. We consider the scalar-type structures; for tensor-type structures, see [12]. The quantum generation process is most easily handled using a perturbed scalar field ($\delta\phi$) equation in the uniform-curvature gauge ($\varphi \equiv 0$), i.e., using $\delta\phi_\varphi$. The analytic forms of power-spectrum of $\delta\phi_\varphi$ are available (in general scales) in variety of inflation stages based on the scalar field and the generalized gravity, [17]. Using Eq. (4) the power-spectrum of $\delta\phi_\varphi$ is directly related to the power-spectrum of $\varphi_{\delta\phi}$, and the latter quantity is conserved during a super-horizon scale evolution which may be the case for the observationally relevant large-scale structures in post inflationary era. As long as the scale remains in the super-horizon it is conserved independently of the reheating process, possible gravity change (e.g., from the generalized gravity to Einstein one), and the change from the scalar field dominated to fluid dominated stages; i.e., $\varphi_{\delta\phi} = C(\mathbf{x}) = \varphi_v$. Later on, in the

hydrodynamic stage, from φ_v we can derive the rest of perturbation variables; afterall, since we are considering the linear perturbation, a variable is a linear combination of the other variables. It is known that φ_χ (φ in the zero-shear gauge which takes $\chi \equiv 0$), δ_v ($\delta \equiv \delta\mu/\mu$ in the comoving gauge), and v_χ (v in the zero-shear gauge) closely correspond to the Newtonian potential fluctuation, density contrast, and velocity fluctuations, respectively, [10,7]. We have [7]:

$$\varphi_\chi = C \left(1 - \frac{H}{a} \int_0^t a dt \right) + \frac{H}{a} d, \tag{24}$$

$$\delta_v = \frac{2k^2}{\mu a^2} \varphi_\chi, \tag{25}$$

$$v_\chi = \frac{k}{a^2} \left(-C \int_0^t a dt + d \right), \tag{26}$$

which are valid for general $p(\mu)$, but assuming $K = 0$ and vanishing stresses. The observed anisotropy of the cosmic microwave background radiation in the large angular scale is also related to C at the last scattering epoch as $\delta T/T = -\frac{1}{5} C$, where we assumed a matter dominated era and ignored the decaying mode. As mentioned, $\tilde{D}$ terms in Eq. (13) is $(k/aH)^2$ higher order compared with $d(\mathbf{x})$ term in these solutions.

Thus, in order to determine the Newtonian quantities (φ_χ, δ_v, and v_χ which provide the initial conditions for later nonlinear evolution stage) what we need is the spatial structures encoded in $C(\mathbf{x})$. From Eqs. (4,16) the power-spectrum of spatial distribution of C, $\mathcal{P}_\mathcal{C}$, can be directly related to the power-spectrum of $\delta\phi_\varphi$ in the inflationary stage as

$$\mathcal{P}_C^{1/2} = \mathcal{P}_{\varphi_v}^{1/2} = \mathcal{P}_{\varphi_{\delta\phi}}^{1/2} = \frac{H}{|\dot{\phi}|} \mathcal{P}_{\delta\phi_\varphi}^{1/2}. \tag{27}$$

The evaluation of quantum fluctuations of $\delta\phi_\varphi$ based on the vacuum expectation value leads to $\mathcal{P}_{\delta\phi_\varphi}$; for example, in near-exponential inflation based on scalar field, in the large-scale and in a simplest vacuum choice, we have $\mathcal{P}_{\delta\phi_\varphi}^{1/2} = H/2\pi$, [18,17]. In this paradigm of large-scale structures generated from quantum fluctuations, we can probe the physics in inflation stage using the observed large-scale structures and the anisotropy in the cosmic microwave background radiation. Recent endeavors for reconstructing and constraining the inflation physics from observation can be found in [19].

Note added: After publication of this work we found a new quantity for hydrodynamic scalar-type perturbation which is conserved considering general K, Λ, and time-varying $p(\mu)$: see [20]

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